



9.1 The determinant

Douglas Wilhelm Harder, LEL, M.Math.

dwharder@uwaterloo.ca

dwharder@gmail.com





Introduction

- In this topic, we will
 - Describe the concept of change of area or volume
 - Describe the idea of orientation
 - See that matrices always change the volume by a fixed ratio
 - This ratio will be called the determinant
 - Find the determinant of simple row-operation matrices
 - Use this to find the determinant of an arbitrary matrix
 - Look at some consequences
 - See some simple formulas for calculating the determinant of small matrices



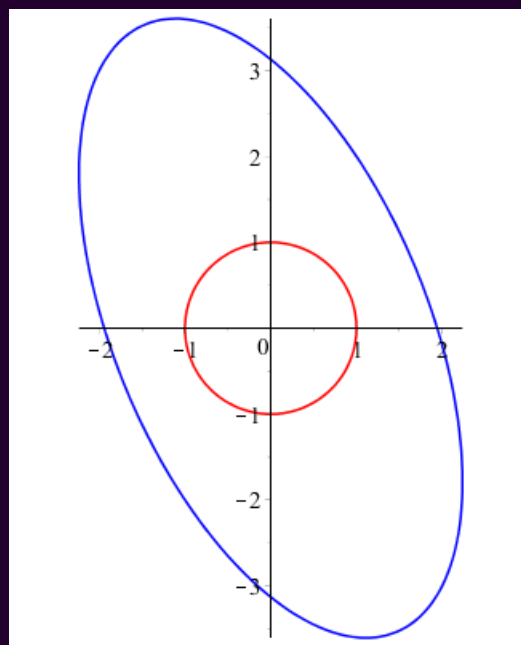
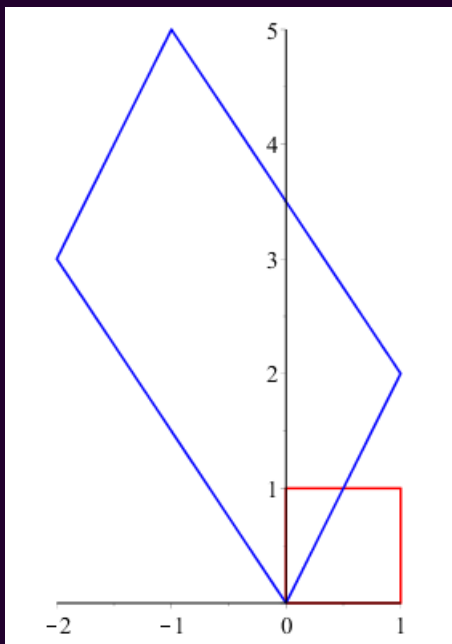


Observation

- If you were to take a square matrix and find the image of some regions,

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix}$$

- We note the ratio of areas seems to be the same





Volumes in $\mathbf{R}^n$

- Suppose that in $\mathbf{R}^n$, we consider the region S defined as all vectors where all entries fall in the real interval $[0, 1]$
 - In $\mathbf{R}^1$, this is a unit line with length 1
 - In $\mathbf{R}^2$, this is a unit square with area 1
 - In $\mathbf{R}^3$, this is a unit cube with volume 1
- In $\mathbf{R}^n$, we will refer to this as a *unit n -cube* with an n -volume of 1





Volumes in $\mathbf{R}^n$

- In $\mathbf{R}^n$, all vectors $\mathbf{u}$ such that $\|\mathbf{u}\|_2 \leq 1$ will be described as the *unit n -ball*

- Its volume can be calculated as

$$\begin{cases} \frac{2^k \pi^k}{(2k)!!} & n = 2k \\ \frac{2^{k+1} \pi^k}{(2k+1)!!} & n = 2k + 1 \end{cases}$$

- Here, $n!! = n(n-2)\cdots 4\cdot 2$ for even n and $n!! = n(n-2)\cdots 3\cdot 1$

- In general, given a region P in $\mathbf{R}^n$, it is possible calculate or estimate its n -volume





Volumes of subspaces

- Notice that:
 - A point in $\mathbf{R}^1$ does not have any length
 - A line or curve in $\mathbf{R}^2$ does not have any area
 - A plane or surface in $\mathbf{R}^3$ does not have any volume
- In general,
a subspace that is not all of $\mathbf{R}^n$ will have zero n -volume





Question on finding the ratio

- Suppose you have a region P in $\mathbf{R}^n$ with an n -volume $\text{vol}(P)$ and you multiply every point in that region by a square matrix $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$
 - Question: Is there a relationship between the $\text{vol}(AP)$ and $\text{vol}(P)$ and can we calculate this ratio?





Effect of row operations

- Let's simplify this problem:
 - Suppose you have a unit n -cube and you multiply every point in that unit n -cube by the matrix $R_{\alpha Rk} : \mathbf{R}^n \rightarrow \mathbf{R}^n$
 - That is, multiply the k^{th} entry by α

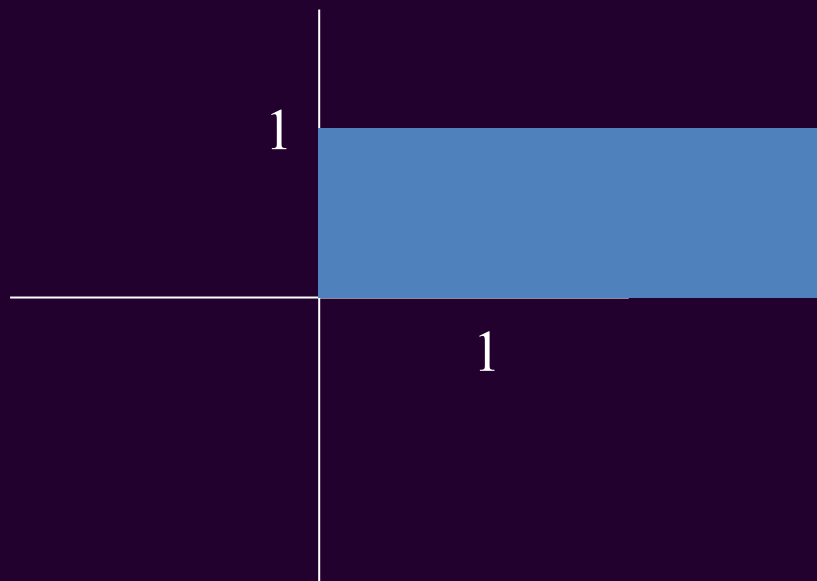




Effect of row operations

- In $\mathbf{R}^2$, we have two options:

$$R_{3R1} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \quad R_{0.5R2} = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}$$





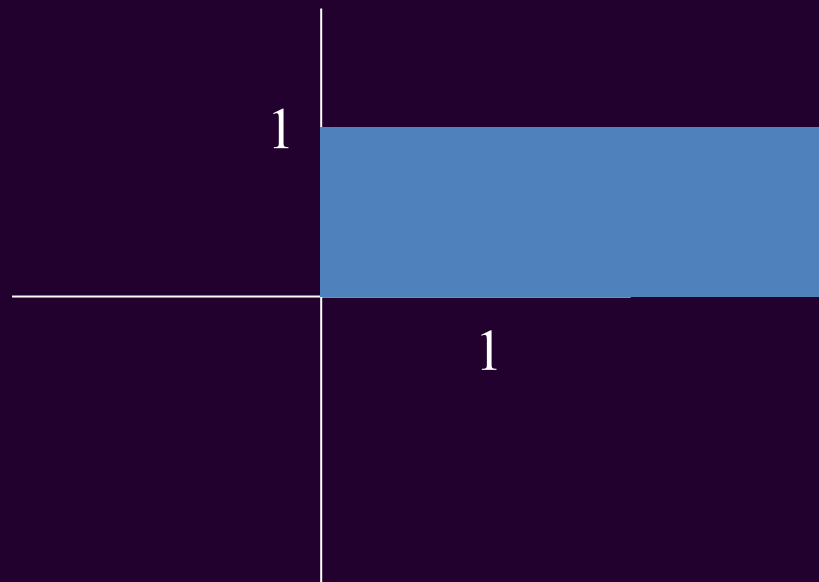
The determinant

- Given an operator A , let us define $\det(A)$ to be the ratio of
 - The n -volume of a region AP
 - The n -volume of the region P

$$\det(R_{\alpha Rk}) = \alpha$$

$$\det(R_{3R1}) = 3$$

$$\det(R_{0.5R2}) = 0.5$$



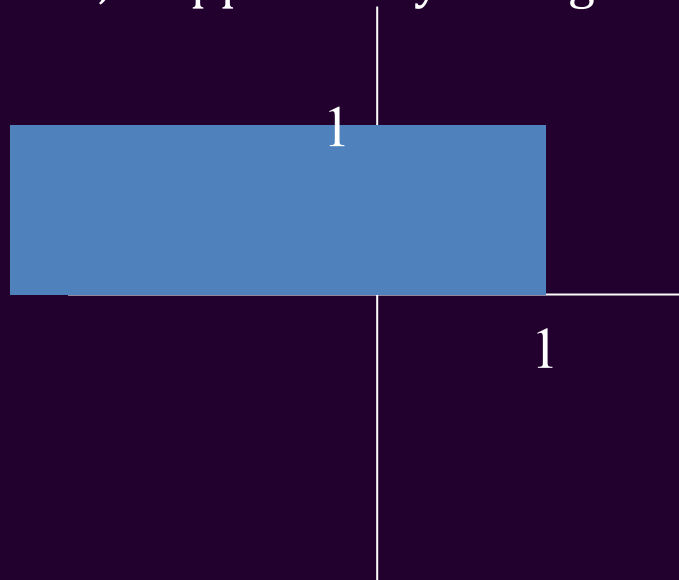


Effect of row operations

- Now, consider the following: what happens if $\alpha < 0$?

$$\det(R_{-2 \cdot R_k}) = -2$$

- Let us interpret a negative number as a flip in the *orientation*
 - When you hold something in your left hand and look at yourself in the mirror, it appears in your right hand



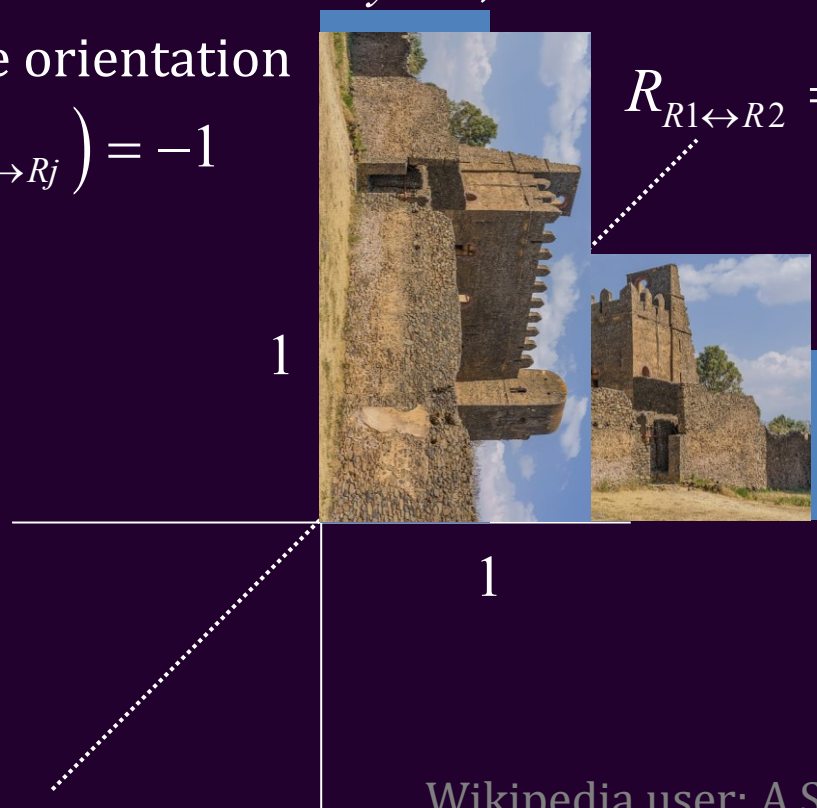


Effect of row operations

- How about swapping two rows:
 - It seems to preserve areas, so $\det(R_{R1 \leftrightarrow R2}) = \pm 1$
 - This is a reflection in the line $y = x$,
so it flips the orientation

$$\det(R_{Ri \leftrightarrow Rj}) = -1$$

$$R_{R1 \leftrightarrow R2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

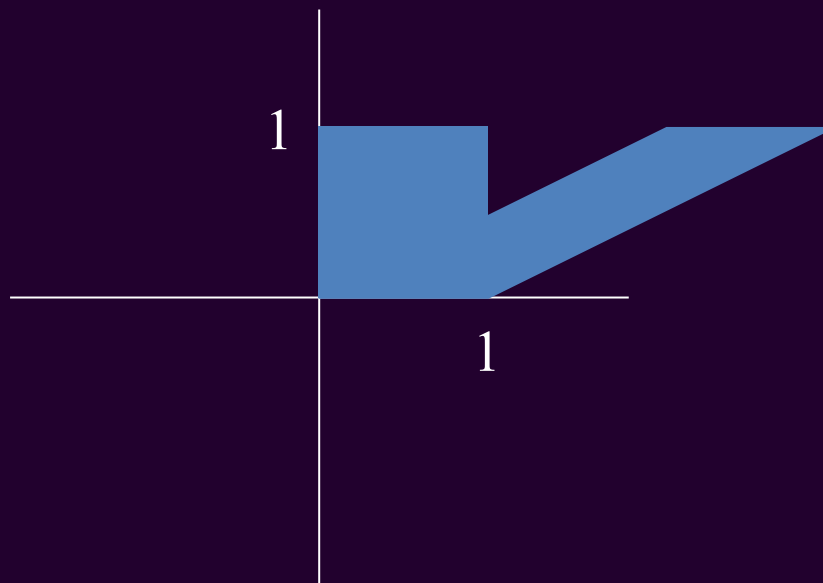




Effect of row operations

- How about adding a scalar multiple of one row onto another?
 - It seems to preserve areas, so $\det(R_{\alpha \cdot R1 \rightarrow R2}) = 1$

$$R_{2R2 \rightarrow R1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

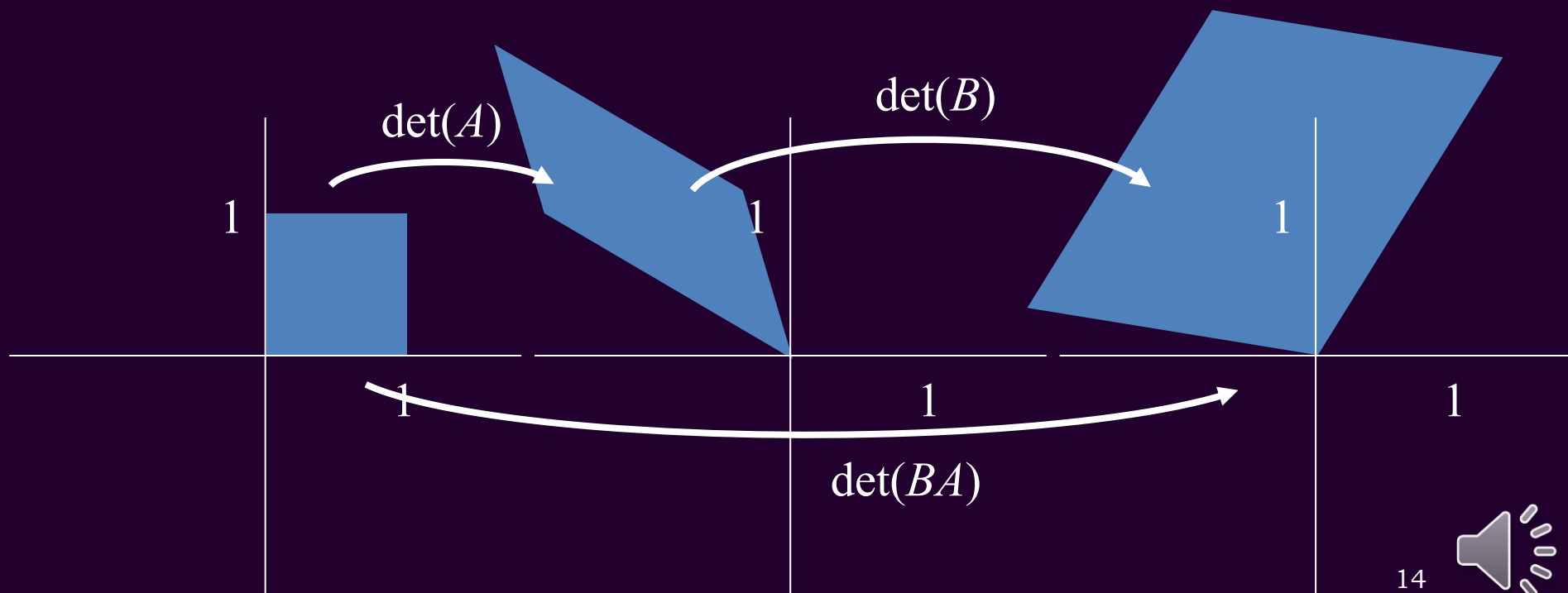




Properties of the determinant

- Now, given $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$ we have defined the determinant $\det(A)$ as the ratio of the change in n -volume under a linear operator A
- Now, given $A, B: \mathbf{R}^n \rightarrow \mathbf{R}^n$, what is $\det(BA)$?

$$\det(BA) = \det(B) \det(A)$$





The identity matrix

Theorem

Given $I_n: \mathbf{R}^n \rightarrow \mathbf{R}^n$, $\det(I_n) = 1$.

Proof:

Given a region P , then $I_n P = P$, so the volumes are the same and there is no change in orientation, thus $\det(I_n) = 1$. ■





Observations

Theorem

Given $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$ and given M row operations

$$R_{\alpha_1 Ri_1 \rightarrow Rj_1}, \dots, R_{\alpha_M Ri_M \rightarrow Rj_M}$$

and $R_{\alpha_M Ri_M \rightarrow Rj_M} \cdots R_{\alpha_1 Ri_1 \rightarrow Rj_1} A = U$, then $\det(A) = \det(U)$.

Proof:

From our previous observation,

$$\det\left(R_{\alpha_M Ri_M \rightarrow Rj_M} \cdots R_{\alpha_1 Ri_1 \rightarrow Rj_1} A\right) = \det(U)$$

$$\text{Thus, } \cancel{\det\left(R_{\alpha_M Ri_M \rightarrow Rj_M}\right)}^1 \cdots \cancel{\det\left(R_{\alpha_1 Ri_1 \rightarrow Rj_1}\right)}^1 \det(A) = \det(U). \blacksquare$$





Observations

- For example,
 - Given the 2×2 identity matrix
 - Add 0.5 times Row 1 onto Row 2
 - Add -3 times Row 2 onto Row 1

$$\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0.5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -0.5 & -3 \\ 0.5 & 1 \end{pmatrix}$$

- Thus, the determinant of the matrix on the right is also 1

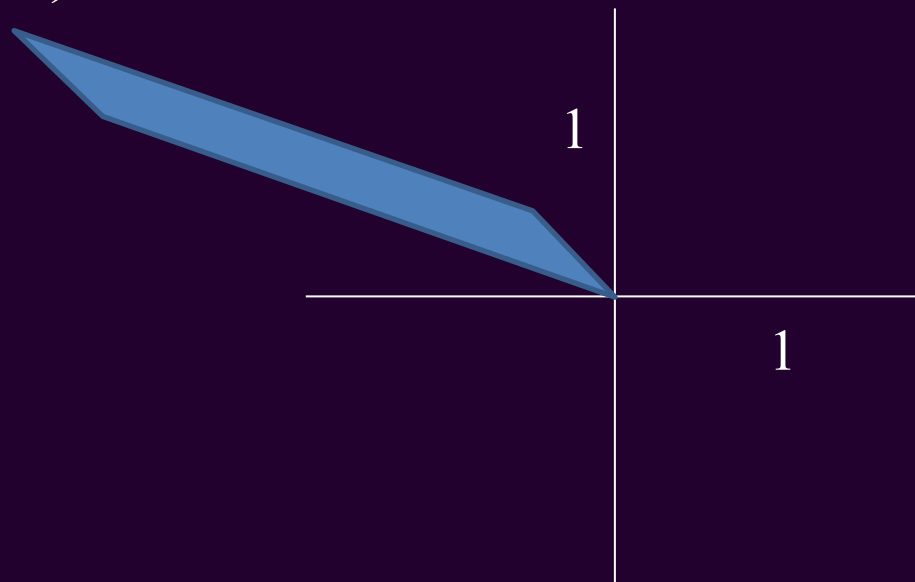
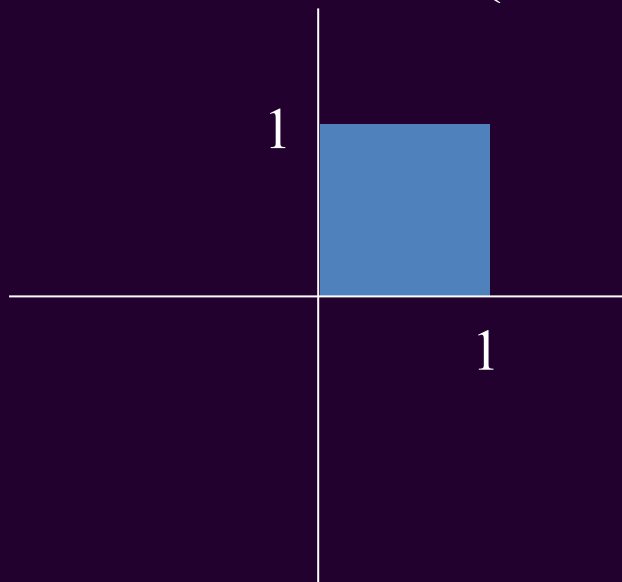




Observations

- Let us check that result

$$\begin{pmatrix} -0.5 & -3 \\ 0.5 & 1 \end{pmatrix}$$





Observations

Theorem

Given $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$ and given $M + N$ row operations

$$R_1, \dots, R_{M+N}$$

where M of these add a multiple of one row onto another,
and N of these swap two rows,

and $R_{M+N} \cdots R_1 A = U$, then $(-1)^N \det(A) = \det(U)$.

Proof:

From our previous observation,

$$\det(R_{M+N} \cdots R_1 A) = \det(U)$$

Thus, $\cancel{\det(R_{M+N})}^{\pm 1} \cdots \cancel{\det(R_1)}^{\pm 1} \det(A) = \det(U)$. ■





Row operations

Theorem

Given $D: \mathbf{R}^n \rightarrow \mathbf{R}^n$ where D is a diagonal matrix, then the determinant is the product of the diagonal entries.

Proof:

A diagonal matrix D is the product of

$$R_{d_{1,1} \cdot R1} \cdot R_{d_{2,2} \cdot R2} \cdot R_{d_{3,3} \cdot R3} \cdots R_{d_{n,n} \cdot Rn}$$

$$\begin{aligned} \det(D) &= \det\left(R_{d_{1,1} \cdot R1} \cdot R_{d_{2,2} \cdot R2} \cdot R_{d_{3,3} \cdot R3} \cdots R_{d_{n,n} \cdot Rn}\right) \\ &= \det\left(R_{d_{1,1} \cdot R1}\right) \det\left(R_{d_{2,2} \cdot R2}\right) \det\left(R_{d_{3,3} \cdot R3}\right) \cdots \det\left(R_{d_{n,n} \cdot Rn}\right) \\ &= d_{1,1} \cdot d_{2,2} \cdot d_{3,3} \cdots d_{n,n} \blacksquare \end{aligned}$$





Observations

- For example,
 - Given the 2×2 diagonal matrix

$$\begin{pmatrix} 3 & 0 \\ 0 & -0.5 \end{pmatrix}$$

- Thus, the determinant of the matrix is -1.5

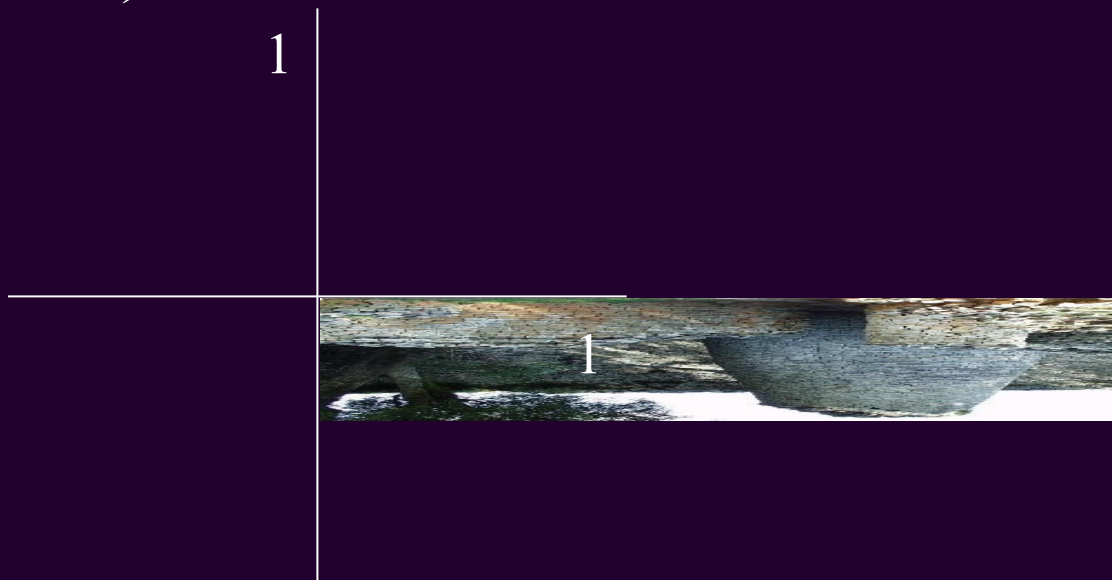
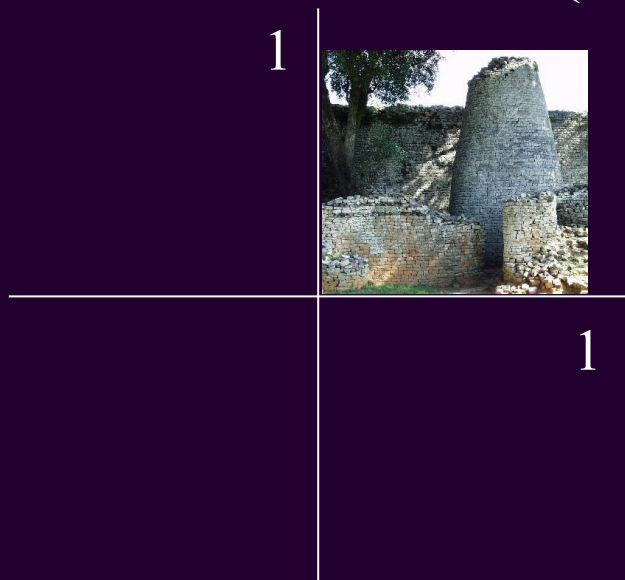




Observations

- Let us check that result

$$\begin{pmatrix} 3 & 0 \\ 0 & -0.5 \end{pmatrix}$$





Row-echelon form

Theorem

Given $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$, if A is in row-echelon form, the determinant is the product of the diagonal entries

Sketch:

If $\text{rank}(A) = n$, then start with the identity matrix.

Multiply each Row k by the non-zero $a_{k,k}$.

Following this, you can add a multiple of subsequent rows onto previous rows to fill in the balance. The first n row operations have determinant $a_{k,k}$ each, and the balance have determinant 1, so the overall determinant is the product of the diagonal entries.





Row-echelon form

Theorem

Given $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$, if A is in row-echelon form, the determinant is the product of the diagonal entries

Sketch:

If $\text{rank}(A) < n$, then to get from the identity to the matrix in row-echelon form, it is necessary to multiply a row by zero, hence the product of the row operation matrices will be zero.

To multiply a row by a zero is a non-invertible operation, and thus further evidence that if the rank of the matrix is less than n , the matrix is not invertible

– You cannot get back to the identity matrix. ■





Row-echelon form

Theorem

Given $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$, if A is in row-echelon form, the determinant is the product of the diagonal entries

Sketch:

If $\text{rank}(A) < n$, we also see the determinant is zero because the image of any region will be a subset of a sub-space of $\mathbf{R}^n$ that is not all of $\mathbf{R}^n$, so if the range has zero n -volume, any subset of this range must also have zero n -volume. ■





Calculating the determinant of a matrix

Theorem

Given $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$, to find the determinant, convert the matrix to row-echelon form, recalling the number of times N that row swaps were performed. Then if U is the row-equivalent matrix in row-echelon form, then

$$\det(A) = (-1)^N u_{1,1} \cdot u_{2,2} \cdot u_{3,3} \cdots u_{n,n}$$

Sketch:

To get from the matrix to the matrix in row-echelon form, it is necessary to apply the row operations of swapping two rows and adding a multiple of one row onto another. The first has determinant -1 and the second 1 . Because the determinant of U is the product of the diagonal entries of a matrix when in row-echelon form, it follows that the determinant of A is $(-1)^N \det(U)$. ■





Observations

Theorem

A matrix $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$ is invertible if and only if the determinant is non-zero.

Proof:

A matrix is invertible if and only if the $\text{rank}(A) = n$, which is true if and only if all diagonal entries of the row-equivalent matrix in row-echelon form are non-zero, which is true if and only if the product of the diagonal entries is non-zero, which is the determinant of A . ■





Observations

Theorem

If a matrix $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$ is invertible, then $\det(A^{-1}) = \det(A)^{-1}$.

Proof:

If A is invertible, then $A A^{-1} = I_n$.

Thus, $\det(A) \det(A^{-1}) = \det(AA^{-1}) = \det(I_n) = 1$.

Thus, $\det(A^{-1}) = \det(A)^{-1}$. ■





Observations

Theorem

If both $A, B: \mathbf{R}^n \rightarrow \mathbf{R}^n$, the matrix BA is invertible if and only if both A and B are invertible.

Proof:

$\det(BA) = \det(B) \det(A)$, which is non-zero if and only if both $\det(A)$ and $\det(B)$ are non-zero. ■





Calculating the determinant

- There are two special cases where it is easy to calculate the determinant:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det \begin{pmatrix} a & b & c & d & e & f \\ d & e & f & g & h & i \\ g & h & i & a & b & c \end{pmatrix}$$

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh - afh - bdi - ceg$$

Critical: This pattern DOES NOT WORK for $\mathbf{R}^4$ or higher.





Numerical stability

- While we have argued that a matrix is invertible if and only if the determinant is non-zero,

do not use this to test for invertibility

- Numerical error means that it is almost guaranteed that the determinant will be non-zero

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 45 + 84 + 96 - 48 - 72 - 105 = 0$$

- In Matlab, the result is 6.6613×10^{-16}





Numerical stability

- Note that a small determinant does not mean a matrix is not perfectly invertible

$$\det \begin{pmatrix} 0.000001 & 0 & 0 \\ 0 & 0.000001 & 0 \\ 0 & 0 & 0.000001 \end{pmatrix} = 10^{-18}$$

- The determinant is small,
but the inverse exists and is easy to calculate:

$$\begin{pmatrix} 0.000001 & 0 & 0 \\ 0 & 0.000001 & 0 \\ 0 & 0 & 0.000001 \end{pmatrix}^{-1} = \begin{pmatrix} 1000000 & 0 & 0 \\ 0 & 1000000 & 0 \\ 0 & 0 & 1000000 \end{pmatrix}$$





Summary

- Following this topic, you now
 - Understand the idea of n -volume in $\mathbf{R}^n$
 - Understand that the row-operation matrices change these n -volumes with simple-to-calculate formulas
 - Understand the idea of the determinant and the idea of orientation
 - Know that the determinant of a product of matrices is the product of the determinants
 - Know how to find the determinant of an upper-triangular matrix, which includes those in row-echelon form
 - Know how to calculate the determinant of an arbitrary matrix
 - Know the simple formulas for calculating the determinants of 2×2 and 3×3 matrices





Warning

- Some of you will search online to find how to calculate the determinant
 - You will come across formulas described as co-factor expansions, and tricks to speed up this horrible algorithm
 - Associated with these are classical adjoints to calculate the inverse
- We cover none of these in this courses, and we shouldn't
 - These formulas have worse run times than almost all other algorithms you will see here at the university
 - They are highly susceptible to numerical error, for calculating the determinant of a 10×10 matrix requires over three million floating-point operations
 - Gaussian elimination requires approximately 1000





References

- [1] https://en.wikipedia.org/wiki/Elementary_matrix
- [2] <https://en.wikipedia.org/wiki/Determinant>





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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